The pro-étale site

@ UIUC chromatic seminar

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Outline

Étale morphisms of adic spaces.

The pro-étale site.

Perfectoid.

Perfectoid form a basis for pro-étale topology.

Étale morphisms of adic spaces

Definition

(i) A morphism of affinoids $(A, A^+) \to (B, B^+)$ is *finite étale* if $A \to B$ is finite étale with the induced topology and B^+ is the integral closure of A^+.

(ii) A morphism $f : X \to Y$ of adic spaces over k is *finite étale* if

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is finite étale.

Étale morphisms of adic spaces

Definition

(iii) A morphism $f : X \to Y$ of adic spaces over k is *étale* if for any point $x \in X$, there are open neighborhoods U and V and a factorization $f|_U = p \circ j$ where j is an open embedding and p is finite étale.

Pro-objects

Let $\mathcal{C}$ be a category.
There is a fully faithful embedding $\mathcal{C} \to \widehat{\mathcal{C}} = Func(\mathcal{C}, Set)^{op}$.

Definition

The category $Pro(\mathcal{C})$ of pro-objects of C is the full subcategory of those objects of $\widehat{C}$ which are small cofiltered inverse limits of representable objects.

$$Pro(\mathcal{C})\Big(F, G\Big) \simeq \varprojlim_{j \in \mathcal{J}} \Big(\varinjlim_{i \in \mathcal{I}} \mathcal{C}(X_i, Y_j) \Big)$$

The pro-étale site

Let X be a rigid-analytic space over K. We define the pro-étale site $X_{pro\acute{e}t}$ as follows,

Definition

$X_{pro\acute{e}t}$ has as underlying category the full subcategory of $Pro(X_{\acute{e}t})$ of objects $U = \varprojlim U_i \to X$ where U_i are rigid-analytic spaces étale over X. We require that $U_i \to U_j$ is finite étale and surjective for $i \gg j$.
A covering is given by a family of pro-étale morphisms $f_i : U_i \to U$ such that $|U| = \cup_i f_i(|U_i|)$.

The pro-étale site

Let X be a rigid-analytic space over K. Note that there is a morphism of sites $v : X_{pro\acute{e}t} \to X_{\acute{e}t}$.
Define
$\mathcal{O}^+ = v^*\mathcal{O}^+_{X_{\acute{e}t}}$ or equivalently, $\mathcal{O}^+(U) = \underrightarrow{lim}\mathcal{O}^+(U_i)$,
$\widehat{\mathcal{O}}^+ = \underleftarrow{lim}\mathcal{O}^+/p^n$,
$\widehat{\mathcal{O}} := \widehat{\mathcal{O}}^+[1/p]$.
We will be interested in the pro-étale cohomology $H^i(X_{pro\acute{e}t}, \widehat{\mathcal{O}}^+)$.

Perfectoid fields

To study it, the use of perfectoid spaces will be crucial.

Definition

A perfectoid field K is a complete topological field whose topology is induced by a nondiscrete valuations of rank 1, such that the Frobenius is surjective on K°/p.

Theorem (Fontaine-Wintenberger)

The absolute Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))$ are canonically isomorphic.

$K = \mathbb{Q}_p\widehat{(p^{1/p^\infty})}$, $K^\flat = \mathbb{F}_p\widehat{(t^{1/p^\infty})}$ are perfectoid fields.

Perfectoid K-algebra

Scholze gave us a geometric generalization of this in adic spaces.

Definition

Let K be a perfectoid field containing $\mathbb{Q}_p$. A topological K-algebra R is perfectoid if the following conditions hold:

R° is bounded in R.

R° is p-adically complete.

There exists a pseudo-uniformizer $\varpi \in R^\circ$ such that $\varpi^p | p$ and the p-power map

$$R^\circ/\varpi \to R^\circ/(\varpi)^p$$

is an isomorphism.

A Huber pair (R, R^+) is a perfectoid affinoid if R is perfectoid. An adic space is perfectoid if it is covered by perfectoid affinoid algebras.

Affinoid perfectoid objects in $X_{pro\acute{e}t}$

Definition

Let X be a rigid-analytic space over K. An object $U \in X_{pro\acute{e}t}$ is *affinoid perfectoid* if $U = \varprojlim Spa(R_i, R_i^+)$, and if $(R, R^+) := (\widehat{\varinjlim R_i}[1/p], \widehat{\varinjlim R_i})$ is perfectoid affinoid K-algebra. An object $U \in X_{pro\acute{e}t}$ is *perfectoid* if it admits an open cover by perfectoid affinoid objects.

Affinoid perfectoid objects in $X_{proét}$

Theorem (Scholze, Colmez)

Let X be a rigid-analytic space over K. Then the set of $U \in X_{proét}$ which are affinoid perfectoid form a basis for the topology.

The theorem works for all locally noetherian adic space over K.

Proof.

On the board. □

References

Scholze, Perfectoid spaces,
p-adic hodge theory for rigid-analytic varieties (Proposition 4.8)