Motivic Tom Dieck Splitting

Ph.D. Defense

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Background and Motivation

- The tom Dieck Splitting (1970s). *Transformation Groups and Representation Theory, 1979*
- The Adams Isomorphism (1984).
- Generalization to compact Lie groups (1986). *Equivariant Stable Homotopy Theory*

$$(Tw \otimes -)_G \simeq (-)^G$$

- Algebraic groups?

Background and Motivation

$$
\begin{tikzcd}
\{\textbf{Compact Lie Groups}\} \arrow[d, shift left, "complexification"] \\
\{\textbf{Linearly Reductive Groups}\} \arrow[u, shift left, "-(\mathbb{C})"]
\end{tikzcd}
$$

G linearly reductive over k. $BG = [S/G]$.

- Genuine G-motivic spectra $\mathcal{SH}(BG)$.
- Homotopy G-motivic spectra $\mathrm{SH}(BG)$.
  Monoidal unit $\mathbf{1}_{BG}$ is NOT compact.

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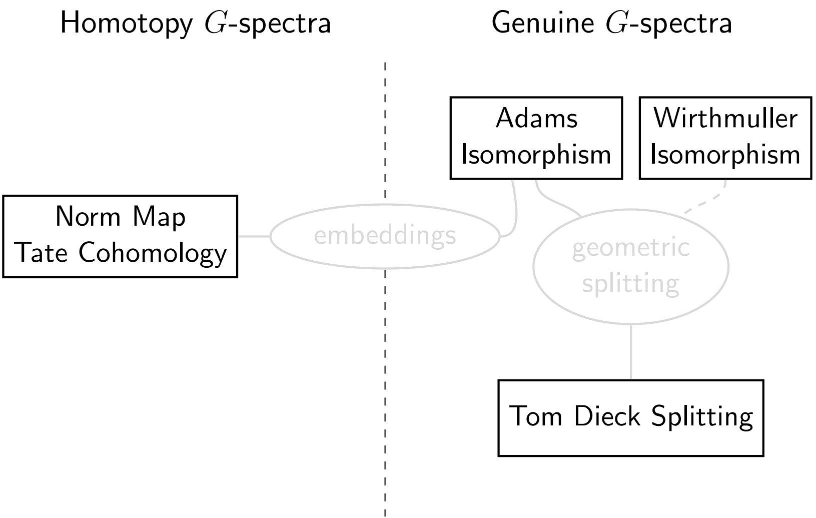
Road Map



Homotopy G-spectra
Genuine G-spectra
Adams Isomorphism
Wirthmuller Isomorphism
Norm Map
Tate Cohomology
embeddings
geometric splitting
Tom Dieck Splitting

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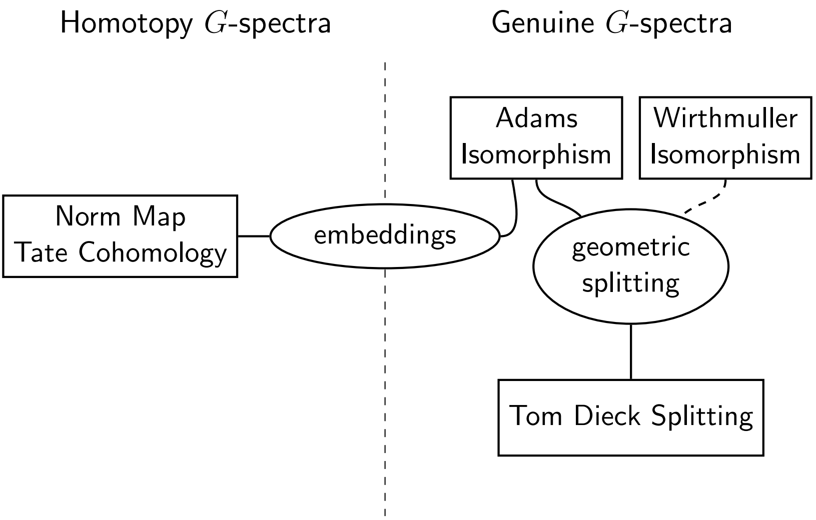

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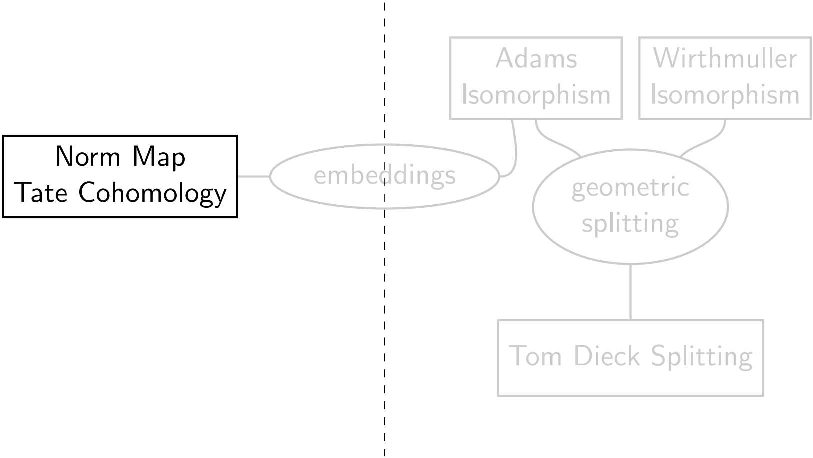
Norm Map And Tate Cohomology

Homotopy G-spectra   Genuine G-spectra



Norm Map And Tate Cohomology

Throughout this section, $\mathcal{X} = BG = [S/G]$.

$$
\begin{tikzcd}
\mathcal{X} \arrow[dr, "\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& \mathcal{X} \times_S \mathcal{X} \arrow[r, "q_1"'] \arrow[d, "q_2"'] & \mathcal{X} \arrow[d, "f"] \\
& \mathcal{X} \arrow[r, "f"'] & S
\end{tikzcd}
$$

$$Nm : f_\sharp(- \otimes \mu_f) \to f_\ell(-) : \mathrm{SH}(BG) \to \mathrm{SH}(S)$$

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$$Nm : f_\sharp(- \otimes \mu_f) \to f_\ell(-) : \mathrm{SH}(BG) \to \mathrm{SH}(S)$$

Norm Map And Tate Cohomology

Theorem (F.)

The norm map

$$Nm : f_\sharp(- \otimes \mu_f) \to f_\ell(-)$$

is an equivalence as functors from $\mathrm{SH}(BG)$ *to* $\mathrm{SH}(S)$*.*

Norm Map And Tate Cohomology

Definition

The Tate fixed point $(-)^{tG} : \mathrm{SH}(BG) \to \mathrm{SH}(S)$ is the cofiber of the map $f_\ell \to f_*$.

Corollary

If G is special, then $(-)^{tG}$ vanishes on all induced objects. Therefore, $(-)^{tG}$ is lax symmetric monoidal.

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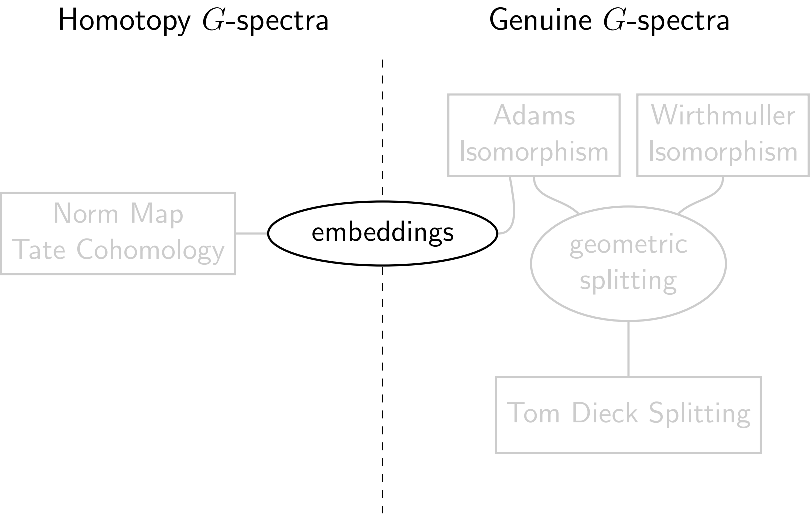
Embeddings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Embeddings

$$\begin{tikzcd}
\mathrm{SH}(BG) \arrow[rr, bend left, "\iota_\sharp"] \arrow[rr, bend right, "\iota_*"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] & & \mathcal{SH}(BG) \arrow[ll, "\iota^*"']
\end{tikzcd}$$

Embeddings

Proposition

If X is a compact object in $\mathrm{SH}(BG)$, then

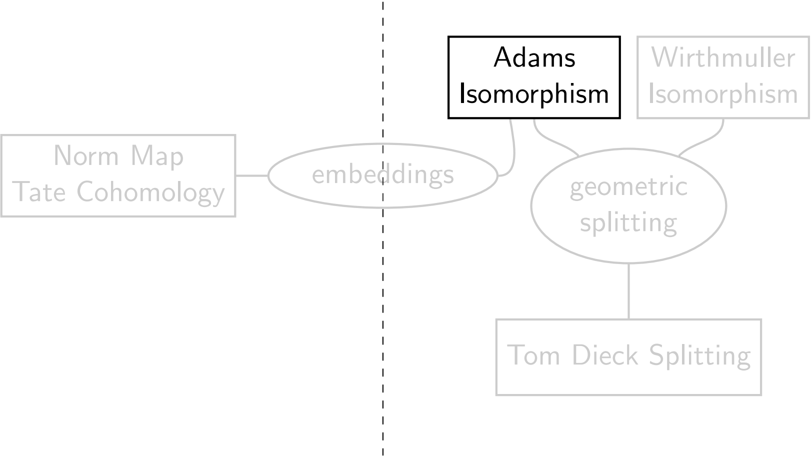
$$\iota_\sharp X \simeq \iota_* X$$

.

Motivic Adams Isomorphism

Homotopy G-spectra

Genuine G-spectra



Motivic Adams Isomorphism

$$\begin{tikzcd}[column sep=huge]
\mathrm{SH}(BG) \arrow[r, bend left=50, "\iota_\sharp"{name=A}] \arrow[r, bend right=50, "\iota_*"'{name=B}] & \mathcal{SH}(BG) \arrow[l, "\iota^*"{name=C}] \arrow[r, bend left=50, dashed, "\pi_\sharp"{name=D}] \arrow[r, bend right=50, "\pi_*"'{name=E}] & \mathrm{SH}(S) \arrow[l, "\pi^*"{name=F}]
\arrow[from=A, to=C, phantom, "\perp"] \arrow[from=C, to=B, phantom, "\perp"]
\arrow[from=D, to=F, phantom, "\perp"] \arrow[from=F, to=E, phantom, "\perp"]
\end{tikzcd}$$

Theorem (F.)

For any object X in $\mathrm{SH}(BG)$, the motivic Adams transformation

$$\pi_\sharp(\iota_\sharp(X \otimes \mu_f)) \to \pi_*(\iota_\sharp X)$$

is an equivalence, where $\mu_f := p_{1*}\Delta_\sharp \mathbf{1}_{BG}$ in $\mathrm{SH}(BG)$.

Motivic Adams Isomorphism

$$\begin{tikzcd}[column sep=huge]
\mathrm{SH}(BG) \arrow[r, bend left=40, "\iota_\sharp"] \arrow[r, bend right=40, "\iota_*"'] & \mathcal{SH}(BG) \arrow[l, "\iota^*" description] \arrow[r, dashed, bend left=40, "\pi_\sharp"] \arrow[r, bend right=40, "\pi_*"'] & \mathrm{SH}(S) \arrow[l, "\pi^*" description]
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Isotropy Filtration

Definition

A G-affine bundle $X \to Y$ is a G-morphism that is a torsor under a G-vector bundle V over X, such that the V action is G-equivariant.

- G acts via affine maps.
- No 'zero section'.
- Not Zariski locally trivial.

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Isotropy Filtration

Definition

Let X be a G-scheme. The *reductive core* of G_x, denoted by G_x^r, is the quotient group $G_x/R_u(G_x)$.

$\mathrm{stab}(X)$: conjugacy classes of G_x^r

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Isotropy Filtration

Definition

Let $E\mathcal{F}$ be the object in $\mathrm{Pre}(\mathrm{Sm}_{BG})$ defined by

$$E\mathcal{F}(X) = \begin{cases} \mathbf{1} & \text{if } \mathrm{stab}(X) \subseteq \mathcal{F} \\ * & \text{if otherwise} \end{cases}$$

Proposition

$$E\mathcal{F} \in \mathcal{H}(S)$$

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Isotropy Filtration

Proposition

The prestratification on $\mathcal{SH}(BG)$ *is a symmetric monoidal stratification. And the map* $\widetilde{\Phi}_G^H : \mathcal{SH}(BG)_H \to \mathrm{SH}(BW(H))$ *is an equivalence.*

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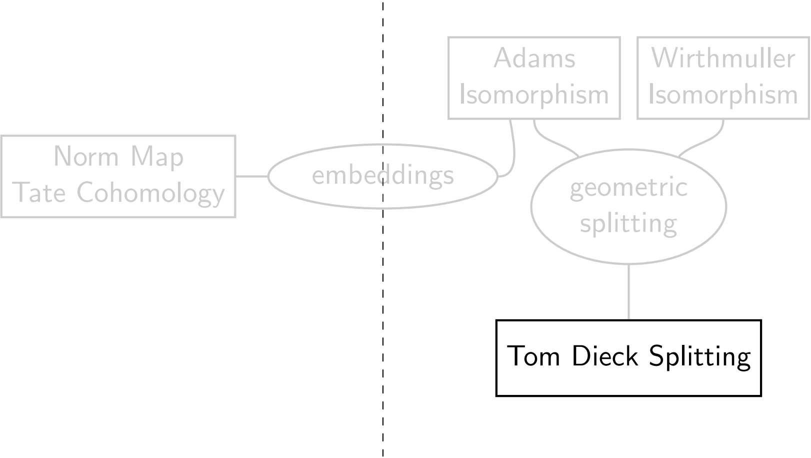
Homotopy G-spectra

Genuine G-spectra



Motivic Tom Dieck Splitting

Let $j : K \to G$ be the inclusion of the subgroup K. We have the following adjunction pairs.

$$\begin{tikzcd}
\mathcal{SH}(BK) \arrow[r, bend left=50, "j_\sharp"] \arrow[r, shift right=2, "j_*"'] & \mathcal{SH}(BG) \arrow[l, shift right=2, "j^*"'] \arrow[l, bend left=50, "j^\sharp"]
\end{tikzcd}$$

Full Wirthmuller isomorphism

$$j_\sharp(- \otimes \nu) \simeq j_*(-)$$

is not available.

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Partial Wirthmuller isomorphism:

Lemma

If X is a dualizable object in $\mathcal{SH}(BK)$ and $j_\sharp X$ is dualizable in $\mathcal{SH}(BG)$, we have

$$j_\sharp X \otimes Y \simeq j_*(X \otimes j^\sharp Y)$$

Motivic Tom Dieck Splitting

Assume that for all (reductive) subgroups H of G.

$$\begin{tikzcd} (j^*X)^H \arrow[r] & (j^*X)^{\Phi_H} \arrow[l, bend right, "s_H"'] \end{tikzcd}$$

$$\psi_H : (E_{WH} \otimes (j_H^* X)^{\Phi H})^{WH} \to X^G$$

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Motivic Tom Dieck Splitting

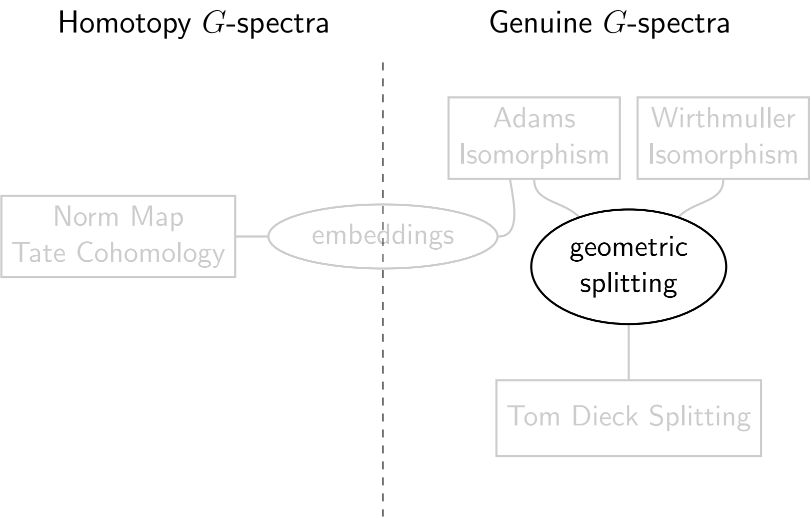
Theorem (F.)

Let X be G-split motivic spectrum in $\mathcal{SH}(BG)$. Then there exists an equivalence

$$\Psi : \bigoplus_{\{H\}\in P_G} (X^{\Phi H} \otimes \mu_{WH})_{hWH} \to X^G$$

where μ_{WH} is μ_f for the projection map $f : BWH \to S$.

Splitting of the Stratification



Splitting of the Stratification

Theorem (F.)

Let P be an Artinian poset and let X be a F-split object in $\mathscr{X}$. Then

$$\Omega : \bigoplus_{p \in P} F(\mathcal{L}_p X) \xrightarrow{\simeq} F(X)$$

is an equivalence.

Corollary

Motivic tom Dieck splitting for commutative groups (of multiplicative type).

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Thank You