An introduction to equivariant motivic homotopy theory

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October 22, 2021

1 Equivariant geometry

1.1 Group scheme action

In this section, we assume B is an arbitrary base scheme and G is a flat finitely generated group scheme over B.

Definition . Given an action $\sigma : G \times S \to S$, *a quasi-coherent G-module on S* is a quasi-coherent sheaf $\mathcal{M}$ on S, together with the isomorphism of $\mathcal{O}_{G\times S}$-modules:

$$\phi : \sigma^*\mathcal{M} \to p_2^*\mathcal{M}$$

that satisfies the cocycle condition. We denote by $\mathrm{QCoh}^G(S)$ the category of quasi-coherent G-modules on S.

Remark . We may view this as two compatible actions: the action σ on the scheme and the action on the sheaf $\mathcal{M}$. For example, if $\mathcal{M}$ is a vector bundle over S, a G-module $\mathcal{M}$ is a G-action on $\mathcal{M}$ which covers σ.

Definition . The G-fixed-point functor $(-)^G : \mathrm{QCoh}^G(B) \to \mathrm{QCoh}(B)$ is the functor π_*, in which $\pi : G \to B$ is the structure map.

1.2 Quasi-affine and Quasi-projective morphisms

Definition . Let $f : X \to Y$ be a G-morphism.
(1) G-quasi-affine (resp.G-affine) if there exists a locally free G-module $\mathcal{E}$ on Y and a quasi-compact G-immersion (resp. a closed G-immersion) $X \hookrightarrow \mathbb{V}(\mathcal{E})$.
(2) G-quasi-projective (resp.G-projective) if there exists a locally free G-module $\mathcal{E}$ on Y and a quasi-compact G-immersion (resp. a closed G-immersion) $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

Definition . Let X be a G scheme. We say that X has the *G-resolution property* if, for every generated quasi-coherent G-module $\mathcal{M}$ on X, there exists a locally free G-module of finite rank $\mathcal{E}$ and an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{M}$.

Definition . A group scheme G over B is called *linearly reductive* if the G-fixed-point functor $(-)^G : \mathrm{QCoh}^G(B) \to \mathrm{QCoh}(B)$ is exact.

We will use linear reductivity only through the through the following lemma.

Lemma . Suppose that B is affine and that G is linearly reductive, and let $p : S \to B$ be an affine G-morphism. If M is a locally free G-module of finite rank on S, then M is projective in $\mathrm{QCoh}^G(S)$.

Proof. Need to show that $\mathrm{Hom}(\mathcal{M}, -) \cong \Gamma_B \circ (-)^G \circ p_* \circ \mathcal{H}om(\mathcal{M}, -)$ is exact. But this is true by checking the exactness of each functor. □

Definition . Let X be a G-scheme. A *G-affine bundle* is a G-morphism $Y \to X$ that is a torsor under a G vector bundle V over X, such that the action $V \times_X Y \to Y$ is G-equivariant.

1.3 Tame group scheme

Definition . A flat finitely presented group scheme G over B is called *tame* if the following conditions hold:

- B admits a Nisnevich covering by schemes having the G-resolution property.
- G is linearly reductive.

Example . G is tame in the following cases:
(1) B arbitrary and G is finite locally free of order invertible on B.
(2) B arbitrary and G is of multiplicative type. (e.g., $\mathbb{G}_m$ or μ_n)
(3) B has characteristic zero and G is reductive. (e.g. GL_n)

2 Unstable equivariant motivic homotopy theory

In this section, we define the unstable equivariant motivic homotopy ∞-category $H^G(S)$ associated with a G-scheme S.

Fix a qcqs base scheme B and a tame group scheme G over B. We denote by Sch_B^G the category of G-schemes that are finitely presented over B and are Nisnevich-locally G-quasi-projective. If $S \in \mathrm{Sch}_B^G$, we denote by Sch_S^G the slice category over S. The category Sm_S^G is the full subcategory of Sch_S^G spanned by smooth S-schemes.

$\mathcal{P}(\mathrm{Sm}_S^G) = \mathrm{Fun}(\mathrm{Sm}_S^G, \mathit{Spc})$ is the ∞-category of presheaves on Sm_S^G valued in *Spc*.

Now we introduce two important subcategoies of the presheaves, i.e., the presheaves that are homotopy invariant and Nisnevich excisive. We will use then to construct $H^G(S)$ and the localization functors.

2.1 Homotopy invariance and Nisnevich excision

Definition . A presheaf F on Sm_S^G is called *homotopy invariant* if every G affine bundle $Y \to X$ in Sm_S^G induces an equivalence $F(X) \cong F(Y)$. We denote by $\mathcal{P}_{htp}(\mathrm{Sm}_S^G) \subset \mathcal{P}(\mathrm{Sm}_S^G)$ the full subcategory spanned by the homotopy invariant presheaves.

The localization functor $L_{htp} : \mathcal{P}(\mathrm{Sm}_S^G) \to \mathcal{P}_{htp}(\mathrm{Sm}_S^G)$ is the left adjoint of the inclusion. L_{htp} is locally cartesian and preserves finite products [Hoy17, Corollary 3.5].

Definition . Let X be a G-scheme. A *Nisnevich square* over X is a cartesian square

$$\begin{array}{ccc} W & \hookrightarrow & V \\ \downarrow & & \downarrow p \\ U & \overset{i}{\hookrightarrow} & X \end{array}$$

of G schemes where i is an open G immersion, p is étale, and p induces an isomorphism $V \times_X Z \cong Z$, where Z is the reduced closed complement of U in X.

Definition . Let S be a G-scheme. A presheaf F on Sm_S^G is called *Nisnevich excisive* if:

- $F(\emptyset)$ is contractible.
- for every Nisnevich square Q in Sm_S^G, $F(Q)$ is cartisian.

We denote by $\mathcal{P}_{Nis}(\mathrm{Sm}_S^G) \subset \mathcal{P}(\mathrm{Sm}_S^G)$.

Definition . Nisnevich topology on Sm_S^G is the coarsest topology for which:

- the empty sieve covers $\emptyset$.
- for every Nisnevich square in Sm_S^G, $\{U \xrightarrow{i} X, V \xrightarrow{p} X\}$ generates a covering sieve.

$L_{Nis} : \mathcal{P}(\mathrm{Sm}_S^G) \to \mathcal{P}_{Nis}(\mathrm{Sm}_S^G)$ is the localization functor, which is also the sheafification with respect to Nisnevich topology.

2.2 Equivariant motivic spaces

Definition . Let S be a G scheme. A *motivic G-space* over S is a presheaf on Sm_S^G that is both homotopy invariant and Nisnevich excisive. We denote by $\mathrm{H}^G(S) \subset \mathcal{P}(\mathrm{Sm}_S^G)$ the full subcategory of motivic G-spaces over S.

If $f : T \to S$ is a G-morphism, then the functor

$$f_* : \mathcal{P}(\mathrm{Sm}_T^G) \to \mathcal{P}(\mathrm{Sm}_S^G),\ \ f_*(F)(X) = F(X \times_S T)$$

induces a functor $f_* : \mathrm{H}^G(T) \to \mathrm{H}^G(S)$. It admits a left adjoint f^* as f_* preserves limits.

If $f : T \to S$ is a smooth G-morphism, we can identify f^* as precomposition with the forgetful functor. Then f^* itself has a left adjoint $f_\sharp$, which is the left Kan extension along the forgetful functor.

Proposition (Smooth projection formula) **.** Let $f : Y \to X$ be a smooth G-morphism and let $B \to A$ be a morphism in $\mathrm{H}^G(X)$. For every $C \in \mathrm{H}^G(Y)_{/f^*A}$ and every $D \in \mathrm{H}^G(X)_{/B}$, the canonical maps

$$f_\sharp(f^*B \times_{f^*A} C) \to B \times_A f_\sharp C$$

$$f^*\mathrm{Hom}_A(B, D) \to \mathrm{Hom}_{f^*A}(f^*B, f^*D)$$

are equivalences in $\mathrm{H}^G(X)$ and $\mathrm{H}^G(Y)$, respectively.

Proof. It suffices to prove the first equivalence. Since L_{mot} is locally cartesian, it follows from the projection formula of presheaves. □

Theorem (Glueing) **.** Let $i : Z \in S$ be a closed G-immersion with open complement $j : U \hookrightarrow S$. Then, for every $F \in \mathrm{H}^G(S)$, the square

$$\begin{array}{ccc} j_\sharp j^* F & \xrightarrow{\epsilon} & F \\ \downarrow{!} & & \downarrow{\eta} \\ U & \xrightarrow{!} & i_* i^* F \end{array}$$

is cocartesian, where ! denotes a unique map.

Corollary **.** Let $i : Z \in S$ be a closed G-immersion. Then the functor $i_* : textH^G(Z) \to textH^G(S)$ is fully faithful.

2.3 Unstable ambidexterity

Definition **.** A pointed motivic G space over S is a motivic G-space over S equipped with a section $S \to X$. We denote by $\mathrm{H}^G_\bullet(S)$ the ∞-category of pointed motivic G-spaces, i.e., the undercategory $\mathrm{H}^G(S)_{S/}$. We denote by $(-)_+ : \mathrm{H}^G(S) \to \mathrm{H}^G_\bullet(S)$ the left adjoint to the forgetful functor.

The results in Section 2.2 have analogs in $\mathrm{H}^G_\bullet(S)$. In particular, the functors f_*, f_*, $f_\sharp$ lifts to the pointed category.

Proposition **.** Let $i : Z \hookrightarrow S$ be a closed G-immersion. Then the functor $i_* : \mathrm{H}^G_\bullet(Z) \to \mathrm{H}^G_\bullet(S)$ preserves colimit.

By the above proposition, i_* has a right adjoint $i^! : \mathrm{H}^G_\bullet(S) \to \mathrm{H}^G_\bullet(Z)$.

Proposition (Pointed glueing) **.** Let $i : Z \hookrightarrow S$ be a closed G-immersion with open complement $j : U \to S$. For every $X \in \mathrm{H}^G_\bullet(S)$,

$$j_\sharp j^* X \to X \to i_* i^* X$$

is a cofiber sequence, and

$$i_* i^! X \to X \to j_* j^* X$$

is a fiber sequence.

Proof. □

Corollary (Closed projection formula) **.** Let $i : Z \hookrightarrow S$ be a closed G-immersion and let $A \in \mathrm{H}^G_\bullet(S)$. For every $B \in \mathrm{H}^G_\bullet(Z)$ and every $C \in \mathrm{H}^G_\bullet(S)$, the canonical maps

$$A \otimes i_* B \to i_*(i^* A \otimes B).$$

$$\mathrm{Hom}(i^* A, i^! C) \to i^! \mathrm{Hom}(A, C)$$

are equivalences.

Proof. The first equivalence follows from the pointed glueing and the smooth projection formula. □

We define the suspension and loop space functor as follows:

$$\Sigma^{\mathcal{M}} \colon \mathrm{H}^G_\bullet(S) \rightleftarrows \mathrm{H}^G_\bullet(S) : \Omega^{\mathcal{M}}$$

$$p_\sharp s_* \dashv s^! p^*$$

We denote by $\mathrm{S}^{\mathcal{M}}$ the $\mathcal{M}$-sphere $\Sigma^{\mathcal{M}}\mathbf{1}_S \in \mathrm{H}^S_\bullet(G)$. It follows that $\Sigma^{\mathcal{M}} \simeq \mathrm{S}^{\mathcal{M}} \otimes (-)$.

For every short exact sequence of locally free G-modules

$$0 \to \mathcal{N} \to M \to \mathcal{P} \to 0$$

, there is a canonical equivalence

$$\Psi : \mathrm{S}^{\mathcal{M}} \simeq \mathrm{S}^{\mathcal{N}} \otimes \mathrm{S}^{\mathcal{P}}$$

If $\mathcal{M}$ is a locally free G-module on $X \in \mathrm{Sm}^G_S$, the *Thom space* of $\mathcal{M}$ is defined by

$$\mathrm{Th}_X(\mathcal{M}) = p_\sharp \mathrm{S}^{\mathcal{M}}.$$

Theorem (Unstable ambidexterity) **.** Let $\mathcal{E}$ be a locally free G-module of rank ≥ 1 on S and let

3 Stable equivariant motivic homotopy theory

In this section, we assume that one of the following condition holds:

- G is finitely locally free; or
- B has the G-resolution property.

In general, the definition of $\mathrm{SH}^G(S)$ is harder.

3.1 Equivariant motivic spectra

Given a presentably symmetric monoidal ∞-category $\mathcal{C}$ and a set of objects X in $\mathcal{C}$, there exists a functor:

$$\mathcal{C} \to \mathcal{C}[X^{-1}]$$

with the obvious universal property in the $\mathrm{CAlg}(\mathcal{P}r^{L,\otimes})$. Actually we have an explicit description of $\mathcal{C}[X^{-1}$ as follows:

$$\mathcal{C}[X^{-1}] \simeq \mathrm{Stab}_X(\mathcal{C})$$

as $\mathcal{C}$-modules.

Definition . Let S be a G-scheme with structure map $p : S \to B$. The symmetric monoidal ∞-category of *motivic G-spectra* over S is defined by

$$\mathrm{SH}^G(S) = \mathrm{H}^G_\bullet(S)[p^*(\mathrm{Sph}_B)^{-1}].$$

We denote by

$$\Sigma^\infty \colon \mathrm{SH}^G_\bullet(S) \rightleftarrows \mathrm{SH}^G(S) : \Omega^\infty$$

the canonical adjunction.

We can extend the pullback-pushforward adjunction by the universal properties of $\mathrm{SH}^G(S)$: for every G-morphism $f : T \to S$,

$$f^* \colon \mathrm{SH}^G(S) \rightleftarrows \mathrm{SH}^G(T) : f_*$$

Lemma . For every G-morphism $f : T \to S$, the functor

$$\mathrm{H}^G_\bullet(T) \otimes_{\mathrm{H}^G_\bullet(S)} \mathrm{SH}^G(S) \to \mathrm{SH}^G(T)$$

induced by f^* is an equivalence of symmetric monoidal ∞-categories.

The lemma above shows that we can also extend the adjoint pairs from the previous sections.

If $f : X \to S$ is smooth, there is an $\mathrm{SH}^G(S)$ adjunction:

$$f_\sharp \colon \mathrm{SH}^G(X) \rightleftarrows \mathrm{SH}^G(S) : f^*$$

If $i : Z \to S$ is a closed G-immersion, there is an $\mathrm{SH}^G(S)$ adjunction:

$$i_* \colon \mathrm{SH}^G(Z) \rightleftarrows \mathrm{SH}^G(S) : i^!$$

Remark . The smooth and closed projection formula naturally extend to the stable case.

Remark . In the definition of $\mathrm{SH}^G(S)$, we invert the pullback of the sphere on the base B. This makes easier to define the above functors.
It can be shown that

$$\mathrm{SH}^G(S) = \mathrm{H}^G_\bullet(S)[p^*(\mathrm{Sph}_B)^{-1}] \simeq \mathrm{H}^G_\bullet(S)[Sph_S^{-1}].$$

When G is finite locally free, we also have

$$\mathrm{SH}^G(S) = \mathrm{H}^G_\bullet(S)[\mathrm{S}^{p^*(\mathcal{O}_G)^{-1}}]$$

where $\mathcal{O}_G$ is the regular representation of G.

3.2 Ambidexterity and Atiyah duality

Theorem (Ambidexterity) **.** Let $f : X \to S$ be a smooth proper G-morphism. Then the transformation

$$f_\sharp \Sigma^{-\Omega_f} \to f_* : \mathrm{SH}^G(X) \to \mathrm{SH}^G(S)$$

is an equivalence.

Proof. f is Nisnevich locally G-projective since it is proper. Thus, without loss of generality, we can assume that f is smooth and G-projective. Then apply the unstable ambidexterity. □

Corollary (Proper projection formula) **.** Let $p : Y \to X$ be proper G-morphism. For every $A \in \mathrm{SH}^G(X)$ and $B \in \mathrm{SH}^G(Y)$, the canonical map

$$A \otimes p_* B \to p_*(p^* A \otimes B)$$

is an equivalence.

Proof. p is Nisnevich locally G-projective. So it suffices to prove the corollary for p a closed G-immersion or p smooth and G-projective. In the former case, it follows from the closed projection formula. In the latter case, it follows from the smooth projection formula. Thus the corollary follows from the theorem. □

In the classical stable homotopy theory, for X a compact smooth manifold, the dual of $\Sigma^\infty X_+$ is given by the Thom spectrum $Th(N_X \mathbb{R}^n)$ of its normal bundle $N_X \mathbb{R}^n$. This is called Atiyah duality. We have an analog of this in the stable equivariant motivic homotopy category.

Corollary (Atiyah duality) **.** Let $f : X \to S$ be a smooth proper G-morphism. Then $\Sigma^\infty X_+$ is strongly dual to $f_\sharp \mathrm{S}^{-\Omega_f}$ in $\mathrm{SH}^G(S)$.

Proof. $A \in \mathrm{SH}^G(S)$ is strongly dualizable iff $\mathrm{Hom}(A, \mathbf{1}_S) \otimes B \simeq \mathrm{Hom}(A, B)$ for every B. By the smooth projection formula, $\mathrm{Hom}(f_\sharp \mathbf{1}_S, B) \simeq f_* f^* B$. By the proper projection formula $f_* \mathbf{1}_X \otimes B \simeq f_* f^* B$. Thus $f_\sharp \mathbf{1}_S$ is strongly dualizable, with dual $f_* \mathbf{1}_X$. By the theorem, $f_* \mathbf{1}_X \simeq f_\sharp \mathrm{S}^{-\Omega_f}$. □

Reference:

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